A DUAL CHARACTERIZATION OF THE $\mathcal{C}^1$ HARMONIC CAPACITY AND APPLICATIONS

ALBERT MAS, MARK MELNIKOV, AND XAVIER TOLSA

Abstract. The Lipschitz and $\mathcal{C}^1$ harmonic capacities κ and κ_c in $\mathbb{R}^n$ can be considered as high-dimensional versions of the so-called analytic and continuous analytic capacities γ and α (respectively).

In this paper we provide a dual characterization of κ_c in the spirit of the classical one for the capacity α by means of the Garabedian function. Using this new characterization, we show that $\kappa(E) = \kappa(\partial_o E)$ for any compact set $E \subset \mathbb{R}^n$, where $\partial_o E$ is the outer boundary of E, and we solve an open problem posed by A. Volberg, which consists on estimating from below the Lipschitz harmonic capacity of a graph of a continuous function.

1. Introduction

Let $\mathrm{Lip}^1_{loc}(\mathbb{R}^n)$ be the set of real-valued locally Lipschitz functions (with exponent 1) on $\mathbb{R}^n$ and $\mathcal{C}^1(\mathbb{R}^n)$ the set of real-valued continuously differentiable functions on $\mathbb{R}^n$. If $E \subset \mathbb{R}^n$ is a bounded set and

$$\begin{aligned} U'(E) &= \{\varphi \in \mathrm{Lip}^1_{loc}(\mathbb{R}^n) : \mathrm{supp}\Delta\varphi \subset E, \nabla\varphi(\infty) = 0\},\\ U'_c(E) &= \{\varphi \in \mathcal{C}^1(\mathbb{R}^n) : \mathrm{supp}\Delta\varphi \subset E, \nabla\varphi(\infty) = 0\},\end{aligned}$$

the Lipschitz and $\mathcal{C}^1$ harmonic capacities of E are defined by

$$\begin{aligned} \kappa(E) &= \sup\{\langle 1, \Delta\varphi\rangle : \varphi \in U'(E), \|\nabla\varphi\|_\infty \leq 1\},\\ \kappa_c(E) &= \sup\{\langle 1, \Delta\varphi\rangle : \varphi \in U'_c(E), \|\nabla\varphi\|_\infty \leq 1\},\end{aligned} \tag{1.1}$$

where $\langle f, \Delta\varphi\rangle$ means the action of the compactly supported distribution $\Delta\varphi$ on a smooth function f, and $\|\nabla\varphi\|_\infty$ is the L^∞ norm of the gradient $\nabla\varphi$ with respect to the Lebesgue measure in $\mathbb{R}^n$. The symbol Δ denotes the Laplacian operator in $\mathbb{R}^n$.

In order to deal with the problem of harmonic approximation in the $\mathcal{C}^1$-norm, P. Paramonov introduced in [Pa] the capacities κ and κ_c, and gave a description, in terms of these capacities, of the compact sets $E \subset \mathbb{R}^n$ (with $n \geq 2$) such that any $\mathcal{C}^1$ function harmonic in the interior of E can be approximated in the $\mathcal{C}^1$-norm by harmonic functions in a neighborhood of E.

Date: February 2008.

1991 *Mathematics Subject Classification.* Primary 31B05; Secondary 31B15.

Key words and phrases. Lipschitz and $\mathcal{C}^1$ harmonic capacities, Dirichlet and Neumann problems, orthogonal measures.

The three authors are partially supported by grants 2005SGR-007749 (Generalitat de Catalunya) and MTM2007-62817 (MEC, Spain). Albert Mas is also supported by grant AP2006-02416 (Programa FPU del MEC, Spain).

The capacities κ and κ_c can be understood as high-dimensional versions of the so-called analytic and continuous analytic capacities γ and α (respectively). Recall that, for a compact set $E \subset \mathbb{C}$,

$$\gamma(E) = \sup |f'(\infty)|,$$

where the supremum is taken over all analytic functions $f : \mathbb{C} \setminus E \to \mathbb{C}$ with $|f| \leq 1$ on $\mathbb{C} \setminus E$, and $f'(\infty) = \lim_{z\to\infty} z(f(z) - f(\infty))$. The continuous analytic capacity α has the same definition as γ except that one also requires the functions f to be continuous in $\mathbb{C}$ and $|f| \leq 1$ everywhere.

The analytic capacity was first introduced by L. Ahlfors in [Ah] when he was characterizing the removable compact sets for bounded analytic functions in the plane. The continuous analytic capacity was defined by A. Vitushkin in [Vi] when he dealt with the problem of rational approximation in the uniform norm on compact sets of the plane. Both capacities have been studied by many authors since then (see [Gt] for a nice survey on results related with γ and α, and [Da] or [To1] for more recent results). In particular, there exists a dual characterization of α that can be stated as follows: let $\Omega \subset \mathbb{C}$ be the closure of a bounded domain with smooth boundary. Then,

$$\alpha(\Omega) = \inf \left\{ \frac{1}{2\pi} \int_{\partial\Omega} |h(z)| ds : h \in H^1(\Omega^c), h(\infty) = 1 \right\},$$

where ds denotes the arc length and $H^1(\Omega^c)$ is the Hardy space of functions h analytic in $\Omega^c \cup \{\infty\}$ such that the subharmonic function $|h(z)|$ has a harmonic majorant. It is also proved that the infimum is attained, and the function ψ that solves the extremal problem is called the Garabedian function of Ω; so $\alpha(\Omega) = \frac{1}{2\pi} \int_{\partial\Omega} |\psi(z)| ds$.

A classical way to construct the Garabedian function is to use the Hahn-Banach theorem and the F. and M. Riesz theorem. Observe that the quantity $\alpha(\Omega)$ is the norm of the functional $f \mapsto f'(\infty) = \frac{1}{2\pi} \int_{\partial\Omega} f(z) dz$ on the space of continuous functions outside intΩ that are analytic outside Ω, which is a subspace of the continuous functions on $\partial\Omega$. By the Hahn-Banach theorem one can find a measure μ supported on $\partial\Omega$, orthogonal to the functions analytic outside Ω, and such that

$$\alpha(\Omega) = \frac{1}{2\pi} \int_{\partial\Omega} |dz + d\mu|.$$

The F. and M. Riesz theorem ensures that, in fact, $dz + d\mu(z) = \psi(z) ds$ for an analytic function ψ that solves the extremal problem (see [Gt, section I.4] for more details).

The aim of this paper is to give a dual characterization of the $\mathcal{C}^1$ harmonic capacity κ_c in terms of some "Garabedian function" and to use this "function" to deduce some properties of κ_c and κ. This characterization is stated in theorem 3.3 and it is based on the Hahn-Banach theorem, as can be done for the capacity α.

Unfortunately, the F. and M. Riesz theorem can not be generalized to higher dimensions in the sense that we need, because the capacity κ_c is defined in terms of gradients of harmonic functions and there are examples of measures orthogonal to those gradients which are not absolutely continuous with respect to the surface measure (see [Ma]). This means that, instead

of a “Garabedian function”, we will just have a “Garabedian measure” that minimizes some quantity. We will be able to deduce some geometric properties of that minimal measure by adapting a theorem of B. Gustafsson and D. Khavinson about measures on $\partial\Omega$ orthogonal to harmonic gradients (see theorem 3.1) and by proving a theorem about restrictions to $\partial\Omega$ of orthogonal measures on Ω (see proposition 3.2).

Many properties of the Lipschitz and $\mathcal{C}^1$ harmonic capacities were recently proven. The semiadditivity is a very important one, and it was obtained by A. Volberg in [Vo] for the capacity κ. A little bit later, A. Ruiz de Villa and X. Tolsa proved in [RT] that κ_c is also semiadditive. See [MP] and [Rz] for other interesting properties about these capacities.

We will obtain two new results on the capacity κ from theorem 3.3, namely theorem 4.1 and theorem 4.2. The first one states that $\kappa(E) = \kappa(\partial_o E)$ for any compact set $E \subset \mathbb{R}^n$, where $\partial_o E$ denotes the outer boundary of E (i.e., the boundary of the unbounded component of E^c). This property is obvious for the capacity γ because of the defining conditions, but this is no longer trivial for κ. Evidently, one have $\kappa(E) \geq \kappa(\partial_o E)$. The difficulties appear when one tries to prove the reverse inequality. Observe that, by Gauss formula,

$$\langle 1, \Delta\varphi\rangle = \int_{\partial_o V} \nabla\varphi \cdot \eta d\sigma \tag{1.2}$$

for any $\varphi \in U'(E)$, where V is a sufficiently regular neighborhood of E, and η and $d\sigma$ are the normal outward unit vector and surface measure of ∂V, respectively. Suppose, for simplicity, that E is the closure of a bounded simply connected domain, so $\partial_o E = \partial E$. One can try to prove that $\kappa(E) \leq \kappa(\partial E)$ directly from the definition (1.1) and the identity (1.2). The idea is to modify the functions $\varphi \in U'(E)$ inside E to obtain functions $\widetilde{\varphi} \in U'(\partial E)$ such that $\langle 1, \Delta\varphi\rangle = \langle 1, \Delta\widetilde{\varphi}\rangle$. The problem is that one cannot ensure that the gradients $\nabla\widetilde{\varphi}$ are bounded by 1 in E.

The second new result that we have obtained is theorem 4.2, where we solve an open problem posed by A. Volberg (private communication). The problem can be stated as follows:

Problem 1.1. *Let f be a real continuous function defined on the cube $Q_0 = [0, d]^{n-1} \subset \mathbb{R}^{n-1}$ and let $\Gamma = \{(x, f(x)) \in \mathbb{R}^n : x \in Q_0\}$ be the graph of f. Prove that there exists a constant $C > 0$ depending only on n such that $Cd^{n-1} \leq \kappa(\Gamma)$.*

Note that, if $\operatorname{diam}(\Gamma)$ is comparable to d, problem 1.1 states that $\kappa(\Gamma) \geq C\operatorname{diam}(\Gamma)^{n-1}$. This is a reasonable analogue of an important result in the area of analytic capacity which says that $\gamma(E) \geq \frac{1}{4}\operatorname{diam}(E)$ for any continuum (i.e., compact and connected set) $E \subset \mathbb{C}$. This classical result on γ is a consequence of the 1/4-theorem of Koebe (see [Ga, theorem 2.1 of chapter VIII]), and a real variable proof was first obtained by P. Jones by using the notion of curvature of a measure (see [Pj, Section 3.5]).

One cannot expect this kind of estimates on $\kappa(E)$ for any continuum $E \subset \mathbb{R}^n$, because, for example, a segment in $\mathbb{R}^3$ has zero Lipschitz harmonic capacity. In fact, by using the identity (1.2), it is not difficult to show that $\kappa(E) = 0$ for any compact set $E \subset \mathbb{R}^n$ with zero $(n-1)$-Hausdorff measure.

So, to obtain a reasonable analogue of the estimate of the analytic capacity of a continuum for the capacity κ, one has to restrict himself to continua with positive $(n-1)$-Hausdorff measure or, in an easier way, to graphs of continuous functions.

The structure of the paper is the following. Section 2 is devoted to the preliminaries, where we will talk about vector measures, Lipschitz and $\mathcal{C}^1$ harmonic capacities, and harmonicity at infinity (which includes the exterior Dirichlet and Neumann problems). With these notions, we will be ready to state and prove the dual characterization of κ_c (i.e., theorem 3.3). This will be in section 3. Section 4 is devoted to prove the two announced properties of κ.

2. Preliminaries

In the whole paper, we assume $n \geq 2$. The word *smooth* means *of class* $\mathcal{C}^\infty$, wherever we talk about functions or the boundary of an open set. We write χ_E for the characteristic function of a set $E \subset \mathbb{R}^n$. The letter C will denote a constant which may be different at different occurrences and which is independent of the relevant variables under consideration.

We denote by $\mathcal{C}(E)$ the set of real-valued continuous functions defined on a set $E \subseteq \mathbb{R}^n$, and by $\mathcal{C}(E)^n$ the cartesian product of n spaces $\mathcal{C}(E)$.

Given a $\mathcal{C}^\infty$ orientable manifold M of dimension $d \leq n$ and $k \in \mathbb{N} \cup \{\infty\}$, let $\mathcal{C}^k(M)$ be the set of real-valued differentiable functions in M such that their partial derivatives (with respect to the local coordinates chosen in M) of order less than $k+1$ exist and are continuous functions in M. In case that $\partial M \neq \emptyset$, we can take a system of local coordinates

$$\{U \subset M, y = (y_1, \ldots, y_d)\}$$

such that $U \cap M = y^{-1}(\{x_d \geq 0\})$ and $U \cap \partial M = y^{-1}(\{x_d = 0\})$, for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. For the points $p = y^{-1}(x) \in U \cap \partial M$, the partial derivative of a function $f : y(U) \to \mathbb{R}$ with respect to the coordinate x_d at a point $p = (p_1, \ldots, p_{n-1}, 0) \in y(U)$ is defined by the limit (if it exists)

$$\frac{\partial}{\partial x_d} f\Big|_p = \lim_{t>0,\, t\to 0} \frac{f(p_1, \ldots, p_{n-1}, t) - f(p_1, \ldots, p_{n-1}, 0)}{t}.$$

When $M \subset \mathbb{R}^n$, for any function $\varphi \in \mathcal{C}^1(M)$ we can identify the differential $d\varphi$ with a vector function in $\mathcal{C}(M)^n$, so we say that $\varphi \in \mathcal{C}^1(M)$ if $d\varphi \in \mathcal{C}(M)^n$. Clearly, $d\varphi = \nabla\varphi$ if M is an open set. Notice that, if M is the closure of an open set with smooth boundary, $\varphi|_{\partial M} \in \mathcal{C}^1(\partial M)$ for any $\varphi \in \mathcal{C}^1(M)$, and $d(\varphi|_{\partial M})$ is the tangential part of $d\varphi$ with respect to ∂M.

It is an exercise to check that this definition of $\mathcal{C}^1(M)$ agrees with the classical one for closures of open sets with smooth boundary, i.e., $\mathcal{C}^1(M)$ is the set of continuous functions on M such that their gradients on $\mathrm{int} M$ can be extended to continuous vector functions on M. It also agrees with the definition of $\mathcal{C}^1(M)$ given in [Pa] or [Wh].

Our typical situation will be that M is equal to U, $\overline{U}$ or ∂U, for an open set $U \subset \mathbb{R}^n$ with smooth boundary.

2.1. **Vector measures. The Riesz representation theorem.** Given a subset $E \subset \mathbb{R}^n$ and a vector function $f = (f_1, \dots, f_n) : E \to \mathbb{R}^n$, define

$$\|f\|_E = \sup\{|f(x)| : x \in E\},$$

where

$$|f(x)| = \left(\sum_{i=1}^n (f_i(x))^2 \right)^{1/2}.$$

Clearly, $\mathcal{C}(E)^n$ with the norm $\|\cdot\|_E$ is a Banach space.

Given a bounded linear functional Λ on $\mathcal{C}(E)^n$ and a subspace $\mathcal{F} \subset \mathcal{C}(E)^n$, define

$$\|\Lambda\|_{\mathcal{F}} = \sup\{|\Lambda(f)| : f \in \mathcal{F}, \|f\|_E \leq 1\}.$$

For simplicity, we write $\|f\|$ and $\|\Lambda\|$ instead of $\|f\|_{\mathbb{R}^n}$ and $\|\Lambda\|_{\mathcal{C}(E)^n}$, respectively (when there is no confusion on what is E).

Let $\mathcal{M}(E)$ be the space of finite real Borel measures supported on E and $\mathcal{M}(E)^n$ the cartesian product of n spaces $\mathcal{M}(E)$. The elements of $\mathcal{M}(E)^n$ are commonly called vector measures. For $\mu = (\mu_1, \dots, \mu_n) \in \mathcal{M}(E)^n$, define the variation of μ on a subset $F \subset E$ as

$$|\mu|(F) = \sup \left\{ \sum_{j=1}^m |\mu(F_j)| : F = \biguplus_{j=1}^m F_j, \quad F_j \text{ is } \mu_i\text{-measurable } \forall i, j \right\},$$

where

$$|\mu(F_j)| = \left(\sum_{i=1}^n (\mu_i(F_j))^2 \right)^{1/2}.$$

Finally, define the total variation of μ as $\|\mu\|_E = |\mu|(E)$. It is proved that $|\mu|$ is a positive and finite measure on E (see for example [L2, theorem 3.1 of chapter VII]). It is easily seen that $\|\cdot\|_E$ is a norm on the space $\mathcal{M}(E)^n$.

Any vector measure $\mu \in \mathcal{M}(E)^n$ can be considered as a bounded linear functional $\langle \cdot, \mu \rangle : \mathcal{C}(E)^n \to \mathbb{R}$ by putting

$$\langle f, \mu \rangle = \int f d\mu = \sum_{i=1}^n \int f_i d\mu_i.$$

On the other hand, the Riesz representation theorem (for scalar measures) shows that any bounded linear functional on $\mathcal{C}(E)^n$ can be represented as $\langle \cdot, \mu \rangle$ for some vector measure $\mu \in \mathcal{M}(E)^n$.

The concept of vector measure is widely treated in many text books (see for example [DS] or [DU]). However, usually in the literature, the Riesz representation theorem for vector measures is stated in a slightly different setting. The following theorem contains the detailed arguments to prove the corresponding Riesz representation theorem for our specific setting.

Theorem 2.1 (Riesz representation)**.** *The map $\mu \mapsto \langle \cdot, \mu \rangle$ is an isometric isomorphism of $\mathcal{M}(E)^n$ onto the space of bounded linear functionals on $\mathcal{C}(E)^n$, so $\|\mu\|_E = \|\langle \cdot, \mu \rangle\|$ for all $\mu \in \mathcal{M}(E)^n$.*

Proof. By the previous comments, it is enough to prove that the map $\mu \mapsto \langle\cdot,\mu\rangle$ is isometric to obtain the isomorphism. We have to check that for any $\mu \in \mathcal{M}(E)^n$,

$$\|\mu\|_E = \sup\{|\langle f,\mu\rangle| : f \in \mathcal{C}(E)^n, \|f\|_E \le 1\}.$$

We will see first that $\sup\{|\langle f,\mu\rangle| : f \in \mathcal{C}(E)^n, \|f\|_E \le 1\} \le \|\mu\|_E$. By density, it is enough to prove that $|\langle f,\mu\rangle| \le \|\mu\|_E$ for simple vector functions of the form $f = (\sum_m a_m^1 \chi_{F_m}, \dots, \sum_m a_m^n \chi_{F_m})$, where the sums are finite, the a_m^i's are real numbers, the F_j's are disjoint subsets of E, and $\|f\|_E = \sup_m (\sum_{i=1}^n (a_m^i)^2)^{1/2} \le 1$. By the Cauchy-Schwartz inequality,

$$\begin{aligned}
|\langle f,\mu\rangle| &= \left|\sum_{i=1}^n \sum_m a_m^i \mu_i(F_m)\right| \le \sum_m \left(\sum_{i=1}^n (a_m^i)^2 \sum_{j=1}^n (\mu_j(F_m))^2\right)^{1/2} \\
&\le \sup_m \left(\sum_{i=1}^n (a_m^i)^2\right)^{1/2} \sum_m |\mu(F_m)| \le \|f\|_E |\mu|(E) \le \|\mu\|_E.
\end{aligned}$$

Let us prove now that $\|\mu\|_E \le \sup\{|\langle f,\mu\rangle| : f \in \mathcal{C}(E)^n, \|f\|_E \le 1\}$. Let ν be a positive measure such that μ_i is absolutely continuous with respect to ν for all $i = 1,\dots,n$ (for example $\nu = \sum_{i=1}^n |\mu_i|$, where $|\mu_i|$ is the classical variation of the real measure μ_i). Then $\mu_i = h_i\nu$, where h_i is a ν-measurable function. Observe that, if we put $h = (h_1,\dots,h_n)$, then $\mu = h\nu$ and

$$\|\langle\cdot,\mu\rangle\| = \sup\{|\langle f,h\nu\rangle| : f \in \mathcal{C}(E)^n, \|f\|_E \le 1\} \le \int_E |h| d\nu. \tag{2.1}$$

Consider the ν-measurable vector function g defined by $g(x) = h(x)/|h(x)|$ whenever $h(x) \ne 0$ and $g(x) = 0$ otherwise. Lusin's theorem can be adapted to our situation to prove that given $\varepsilon > 0$ there exists $f_\varepsilon \in \mathcal{C}(E)^n$ with $\|f_\varepsilon\|_E \le \|g\|_E \le 1$ and such that

$$\left|\int_E (g - f_\varepsilon) d\mu\right| < \varepsilon.$$

This implies that

$$\begin{aligned}
\int_E |h| d\nu &= \int_E (g\cdot h) d\nu = \int_E g d\mu \\
&\le \left|\int_E (g - f_\varepsilon) d\mu\right| + \left|\int_E f_\varepsilon d\mu\right| \le \varepsilon + \|\langle\cdot,\mu\rangle\|
\end{aligned}$$

for all $\varepsilon > 0$. This estimate together with (2.1) proves that $\|\langle\cdot,\mu\rangle\| = \int_E |h| d\nu$. So, to prove that $\|\mu\|_E \le \|\langle\cdot,\mu\rangle\|$ it is enough to check that $|\mu(F)| \le \int_F |h| d\nu$ for all $F \subset E$ ν-measurable. By a discrete version of Minkowski's integral inequality,

$$\begin{aligned}
|\mu(F)| &= \left(\sum_{i=1}^n \left(\int_F d\mu_i\right)^2\right)^{1/2} = \left(\sum_{i=1}^n \left(\int_F h_i d\nu\right)^2\right)^{1/2} \\
&\le \int_F \left(\sum_{i=1}^n h_i^2\right)^{1/2} d\nu.
\end{aligned}$$

Therefore, $|\mu(F)| \le \int_F |h| d\nu$, and the theorem is proved. $\square$

2.2. **The Lipschitz and $\mathcal{C}^1$ harmonic capacities.** The fundamental solution ϕ_n for the Laplace equation $\Delta f = 0$ in $\mathbb{R}^n$ is defined by

$$\phi_n(x) = \begin{cases} a_n|x|^{2-n} & \text{if } n > 2, \\ a_n \log|x| & \text{if } n = 2, \end{cases}$$

where a_n is a constant which depends on the dimension n.

We stated the definitions of the Lipschitz and $\mathcal{C}^1$ harmonic capacities κ and κ_c in (1.1). The defining conditions for $U'(E)$ and $U'_c(E)$ imply that the functions $\varphi \in U'(E)$ are harmonic in E^c and take the form $\varphi = \phi_n * \Delta\varphi$+constant, where this last equality is in the sense of distributions; but by the definitions of κ and κ_c we can suppose that, in fact, $\varphi = \phi_n * \Delta\varphi$. Recall that, if T is a compactly supported distribution, then for each $\psi \in \mathcal{C}^\infty(\mathbb{R}^n)$ with compact support, by definition (in view of parity of ϕ_n),

$$\langle \phi_n * T, \psi\rangle = \langle T, \phi_n * \psi\rangle,$$

where $\phi_n * \psi(x) = \int \phi_n(y)\psi(x-y)dm(y)$ and m is the Lebesgue measure on $\mathbb{R}^n$.

Therefore, if we take

$$\begin{aligned} U(E) &= \{\varphi \in \mathrm{Lip}^1_{loc}(\mathbb{R}^n) : \mathrm{supp}\Delta\varphi \subset E, \varphi = \phi_n * \Delta\varphi\}, \\ U_c(E) &= \{\varphi \in \mathcal{C}^1(\mathbb{R}^n) : \mathrm{supp}\Delta\varphi \subset E, \varphi = \phi_n * \Delta\varphi\}, \end{aligned}$$

we can redefine the Lipschitz and $\mathcal{C}^1$ harmonic capacities by

$$\kappa(E) = \sup\{\langle 1, \Delta\varphi\rangle : \varphi \in U(E), \|\nabla\varphi\|_\infty \le 1\},$$
$$\kappa_c(E) = \sup\{\langle 1, \Delta\varphi\rangle : \varphi \in U_c(E), \|\nabla\varphi\|_\infty \le 1\}.$$

2.3. **Harmonicity outside a compact set and at infinity.** Most of this section can be found in [Fo].

Definition 2.2. *For any set $E \subset \mathbb{R}^n \cup \{\infty\}$ we define $E^* = \{x/|x|^2 : x \in E\} \subset \mathbb{R}^n \cup \{\infty\}$. Given a function u defined on a set $E \subset \mathbb{R}^n \setminus \{0\}$, define the Kelvin transform of u by*

$$K_u(x) = |x|^{2-n}u(x/|x|^2), \quad \text{for } x \in E^*.$$

Theorem 2.3. *The Kelvin transform is its own inverse. If $V \subset \mathbb{R}^n \setminus \{0\}$ is an open set, then a function u is harmonic in V if and only if K_u is harmonic in V^*.*

Definition 2.4. *If $E \subset \mathbb{R}^n$ is compact and u is harmonic in E^c, then u is harmonic at ∞ provided K_u has a removable singularity at the origin.*

Theorem 2.5. *Suppose that u is harmonic in E^c, where $E \subset \mathbb{R}^n$ is compact. Then, the following three conditions are equivalent:*

(1) *u is harmonic at ∞.*
(2) *$|u(x)| = o(1)$ as $x \to \infty$ $(n > 2)$, or*
$|u(x)| = o(\log|x|)$ as $x \to \infty$ $(n = 2)$.
(3) *$|u(x)| = O(|x|^{2-n})$ as $x \to \infty$.*

In particular, any function which is harmonic at infinity vanishes at infinity when $n > 2$ and is bounded when $n = 2$.

Theorem 2.6 (Exterior Dirichlet problem). *Let $\Omega \subset \mathbb{R}^n$ be the closure of a bounded domain with smooth boundary. Given $h \in \mathcal{C}(\partial\Omega)$ there exists a unique function $u \in \mathcal{C}(\overline{\Omega^c})$ such that u is harmonic in $\Omega^c \cup \{\infty\}$ and $u|_{\partial\Omega} = h$. If $h \in \mathcal{C}^\infty(\partial\Omega)$, then $u \in \mathcal{C}^\infty(\Omega)$.*

Theorem 2.7 (Exterior Neumann problem). *Let $\Omega \subset \mathbb{R}^n$ be the closure of a bounded domain with smooth boundary and η the outward unit normal vector on $\partial\Omega$. Let $V_1, \ldots, V_m$ be the bounded connected components of Ω^c and V_0 the unbounded one. Let $h \in \mathcal{C}(\partial\Omega)$ such that*

$$\int_{\partial V_i} h d\sigma = 0 \quad \textit{for all } i = 1, \ldots, m.$$

(1) *Assume $n > 2$. Then, there exists a function $u \in \mathcal{C}^1(\overline{\Omega^c})$ such that u is harmonic in $\Omega^c \cup \{\infty\}$ and $(\nabla u \cdot \eta)|_{\partial\Omega} = h$. The function u is unique modulo functions which are constant on each bounded connected component of Ω^c.*

(2) *Assume $n = 2$. Then, there exists a function $u \in \mathcal{C}^1(\overline{\Omega^c})$ such that u is harmonic in $\Omega^c \cup \{\infty\}$ and $(\nabla u \cdot \eta)|_{\partial\Omega} = h$ if and only if*

$$\int_{\partial V_0} h d\sigma = 0.$$

In that case, the function u is unique modulo functions which are constant on each connected component of Ω^c.

In both cases, $u \in \mathcal{C}^\infty(\Omega)$ if $h \in \mathcal{C}^\infty(\partial\Omega)$.

Remark 2.8. Theorem 2.7 corresponds with theorem 3.41 of [Fo]. If we do not have the assumption $\int_{\partial V_0} h d\sigma = 0$ in theorem 2.7(2), we can still find a function $u \in \mathcal{C}^1(\overline{\Omega^c})$ harmonic in Ω^c and such that $(\nabla u \cdot \eta)|_{\partial\Omega} = h$, by looking carefully at the proof of theorem 3.41 in [Fo]. Moreover, u can be taken as

$$u(x) = \int_{\partial V_0} \log|x-y| u_0(y) d\sigma(y)$$

for all $x \in V_0$ and for some $u_0 \in \mathcal{C}(\partial V_0)$ depending on h. But now, u may not be harmonic at infinity (because it may not be bounded) and we cannot ensure uniqueness in Ω^c modulo constant functions. In fact, in proposition 3.35 of [Fo] it is shown that our particular solution u is harmonic at infinity if and only if $\int_{\partial V_0} h d\sigma = 0$.

Lemma 2.9 (Green's formula). *Let $\Omega \subset \mathbb{R}^n$ be the closure of a bounded domain with smooth boundary and η the outward unit normal vector on $\partial\Omega$. Let u and v be harmonic functions in Ω^c, $\mathcal{C}^1$ up to $\partial\Omega$, and such that*

$$|(u(x)\nabla v(x) - v(x)\nabla u(x)) \cdot x| = o(|x|^{2-n})$$

when $|x| \to \infty$. Then,

$$\int_{\partial\Omega} (\nabla u \cdot \eta) v d\sigma = \int_{\partial\Omega} u (\nabla v \cdot \eta) d\sigma.$$

Proof. Let B_R be the ball centered at the origin with radius R, and take $R > M$ such that $\Omega \subset B_{R/2}$. Define $\Omega_R = B_R \setminus \Omega$ and let η denote also the

inward unit normal vector on ∂B_R. By Green's formula on Ω_R,

$$\begin{aligned}\int_{\partial\Omega}(\nabla u\cdot\eta)v\,d\sigma &= \int_{\partial\Omega_R}(\nabla u\cdot\eta)v\,d\sigma-\int_{\partial B_R}(\nabla u\cdot\eta)v\,d\sigma\\ &= \int_{\partial\Omega_R}u(\nabla v\cdot\eta)d\sigma-\int_{\partial B_R}(\nabla u\cdot\eta)v\,d\sigma\\ &= \int_{\partial\Omega}u(\nabla v\cdot\eta)d\sigma+\int_{\partial B_R}(u(\nabla v\cdot\eta)-(\nabla u\cdot\eta)v)d\sigma.\end{aligned}$$

For any R big enough, by the assumption on u and v,

$$\begin{aligned}\left|\int_{\partial B_R}(u(\nabla v\cdot\eta)-(\nabla u\cdot\eta)v)d\sigma\right| &\leq \int_{\partial B_R}|u\nabla v\cdot\eta-v\nabla u\cdot\eta|d\sigma\\ &\leq o(R^{1-n})R^{n-1},\end{aligned}$$

and letting $R\to\infty$ we obtain the desired result. □

Remark 2.10. Let $\Omega\subset\mathbb{R}^n$ be the closure of a bounded open set with smooth boundary. In the next section, we will need to apply Green's formula to pairs of functions φ and u, where $\varphi\in U_c(\Omega)$ and u is a function harmonic in $\Omega^c\cup\{\infty\}$ and continuous up to $\partial\Omega$. For this reason, we give now some estimates on the behavior of φ and u near infinity.

Let $\varphi\in U_c(\Omega)$. By applying Gauss formula, we have that for all $x\in\Omega^c$,

$$\begin{aligned}\varphi(x) &= \int\phi_n(x-y)\Delta_y\varphi(y)dm(y)\\ &= \int_\Omega(\operatorname{div}_y(\phi_n(x-y)\nabla_y\varphi(y))-\nabla_y\phi_n(x-y)\cdot\nabla_y\varphi(y))dm(y)\\ &= \int_{\partial\Omega}\phi_n(x-y)\nabla_y\varphi(y)\cdot\eta(y)d\sigma(y)+\int_\Omega\nabla\phi_n(x-y)\cdot\nabla_y\varphi(y)dm(y),\end{aligned}$$

where η and $d\sigma$ are the outward unit normal vector and surface measure related to $\partial\Omega$.

Assume $n>2$. By computing the derivatives of ϕ_n, it is an exercise to see that any $\varphi\in U_c(\Omega)$ satisfies the statement (3) of theorem 2.5, so it is harmonic at infinity. It is proved in [Fo] (proposition 2.73) that any function u harmonic outside Ω and at infinity satisfies $|\nabla u(x)\cdot x|=O(|x|^{2-n})$, so lemma 2.9 can be applied to the pair φ and u, and Green's formula holds in that case.

The case $n=2$ is a little bit different, because we cannot ensure that a function $\varphi\in U_c(\Omega)$ has the required decay at infinity. We have the estimates $|\varphi(x)|=O(\log|x|)$ and $|\nabla\varphi(x)|=O(|x|^{-1})$ near infinity (and we cannot apply theorem 2.5(2)). For a function u harmonic outside Ω and at infinity, we still have the estimates $|u(x)|=O(1)$ and $|\nabla u(x)\cdot x|=O(|x|^{-1})$, as can be seen in proposition 2.73 of [Fo]. These estimates are not enough to use lemma 2.9, because $|u(x)\nabla\varphi(x)\cdot x|=O(1)$. But, if $u(\infty)=0$, then $|u(x)|=o(1)$ and we can still apply lemma 2.9 in that particular case.

3. The heart of the matter

In the whole section, $\Omega \subset \mathbb{R}^n$ will be the closure of a bounded open set with smooth boundary and η will denote the outward unit normal vector on $\partial\Omega$.

Consider the following normed spaces related to the compact set Ω:

$$\begin{aligned} B(\Omega) &= \{f \in \mathcal{C}(\Omega)^n : f = \nabla\varphi, \varphi \in U_c(\Omega)\}, \\ B(\Omega)^\perp &= \{\mu \in \mathcal{M}(\Omega)^n : \langle f, \mu\rangle = 0 \text{ for all } f \in B(\Omega)\}, \\ bB(\Omega)^\perp &= \{\mu \in B(\Omega)^\perp : \operatorname{supp}\mu \subset \partial\Omega\}, \end{aligned}$$

where $B(\Omega)$ is equipped with the norm $\|\cdot\|_\Omega$ and the orthogonal spaces $B(\Omega)^\perp$ and $bB(\Omega)^\perp$ are equipped with the induced norm from $\mathcal{M}(\Omega)^n$. It is easily seen that the norm in $bB(\Omega)^\perp$ induced by the space $\mathcal{M}(\Omega)^n$ coincides with the norm induced by the space $\mathcal{M}(\partial\Omega)^n$.

Let $A(\Omega)$ be the set of smooth vector fields on $\partial\Omega$. For any $g \in A(\Omega)$, let g_τ be the tangential component of g and $g_\eta \cdot \eta$ the normal one, i.e., $g_\eta = g \cdot \eta$ and $g_\tau = g - g_\eta \eta$. Notice that g_η is a scalar function while g_τ is a vector one. Denote by u_g the unique harmonic extension of g_η to $\Omega^c \cup \{\infty\}$ given by theorem 2.6. Define $A_0(\Omega) = \{g \in A(\Omega) : u_g(\infty) = 0\}$. By theorem 2.5, $A_0(\Omega) = A(\Omega)$ for $n > 2$.

For any $g \in A(\Omega)$, the divergence of g_τ on $\partial\Omega$ can be defined by its action on smooth and compactly supported functions φ as

$$\int_{\partial\Omega} (\operatorname{div} g_\tau)\varphi d\sigma = -\int_{\partial\Omega} g_\tau \cdot \nabla\varphi d\sigma.$$

In the right hand side, we can replace $\nabla\varphi$ by $(\nabla\varphi)_\tau$, which only depends on the values of φ on $\partial\Omega$ (remember that $(\nabla\varphi)_\tau = d(\varphi|_{\partial\Omega})$). This definition agrees with the classical one of divergence in the context of Riemannian manifolds in $\mathbb{R}^n$ (see [Wa]).

The following theorem is a little modification of theorem 3.1 in [GK], where the orthogonal of the space of harmonic gradients inside Ω is studied. In our case, we need the harmonicity outside Ω. Our proof is almost the same as the one of [GK] and, for completeness, we include all the detailed arguments.

Theorem 3.1. *Let $g \in A_0(\Omega)$. Then $g d\sigma \in bB(\Omega)^\perp$ if and only if*

$$\operatorname{div} g_\tau = \nabla u_g \cdot \eta \text{ on } \partial\Omega.$$

Such measures are weak dense in $bB(\Omega)^\perp$, i.e., for all $\mu \in bB(\Omega)^\perp$ there exists a sequence $\{g^m\}_{m\in\mathbb{N}} \subset A_0(\Omega)$ such that $\operatorname{div} g^m_\tau = \nabla u_{g^m} \cdot \eta$ for all m and*

$$\lim_{m\to\infty} \langle f, g^m d\sigma\rangle = \langle f, \mu\rangle$$

for all $f \in \mathcal{C}(\partial\Omega)^n$.

Proof. Let $g \in A_0(\Omega)$ and take $\nabla\varphi \in B(\Omega)$. Then,

$$\begin{aligned} \langle \nabla\varphi, g d\sigma\rangle &= \int_{\partial\Omega} (\nabla\varphi)_\tau \cdot g_\tau d\sigma + \int_{\partial\Omega} (\nabla\varphi \cdot \eta) g_\eta d\sigma \\ &= -\int_{\partial\Omega} \varphi \operatorname{div} g_\tau d\sigma + \int_{\partial\Omega} (\nabla\varphi \cdot \eta) u_g d\sigma. \end{aligned}$$

The pair φ and u_g satisfies the statements of lemma 2.9 by remark 2.10, so

$$\langle \nabla\varphi, gd\sigma\rangle = \int_{\partial\Omega} \varphi(\nabla u_g \cdot \eta - \mathrm{div} g_\tau)d\sigma.$$

This integral vanishes for all φ such that $\nabla\varphi \in B(\Omega)$ if and only if $\mathrm{div}g_\tau = \nabla u_g \cdot \eta$ on $\partial\Omega$.

To prove that such measures $gd\sigma$ are weak* dense in $bB(\Omega)^\perp$ it is enough to prove that if $f \in \mathcal{C}(\partial\Omega)^n$ and $\langle f, gd\sigma\rangle = 0$ for all such g, then there exists $\psi \in U_c(\Omega)$ with $(\nabla\psi)|_{\partial\Omega} = f$. So, consider $f \in \mathcal{C}(\partial\Omega)^n$ with

$$0 = \langle f, gd\sigma\rangle = \int_{\partial\Omega} (f_\tau \cdot g_\tau + f_\eta g_\eta)d\sigma \tag{3.1}$$

for all $g \in A_0(\Omega)$ such that $\mathrm{div}g_\tau = \nabla u_g \cdot \eta$ on $\partial\Omega$. By Hodge's decomposition theorem on the Riemannian manifold $\partial\Omega$ (see [MC, lemma 9.1]), $f_\tau = d\varphi + h_\tau$, where $\varphi \in \mathcal{C}^1(\partial\Omega)$ and h_τ is a tangential vector field on $\partial\Omega$ such that $\mathrm{div}h_\tau = 0$. If we take $g_\eta = 0$ and $g_\tau = h_\tau$ in (3.1), we obtain

$$0 = \langle f, gd\sigma\rangle = \int_{\partial\Omega} (d\varphi \cdot g_\tau + h_\tau \cdot g_\tau)d\sigma = \int_{\partial\Omega} |h_\tau|^2 d\sigma,$$

so $h_\tau = 0$ and $f_\tau = d\varphi$. This implies that

$$\int_{\partial\Omega} f_\tau \cdot g_\tau d\sigma = \int_{\partial\Omega} d\varphi \cdot g_\tau d\sigma = -\int_{\partial\Omega} \varphi \mathrm{div} g_\tau d\sigma = -\int_{\partial\Omega} \varphi(\nabla u_g \cdot \eta)d\sigma,$$

and then (3.1) takes the form

$$\int_{\partial\Omega} f_\eta u_g d\sigma = \int_{\partial\Omega} \varphi(\nabla u_g \cdot \eta)d\sigma. \tag{3.2}$$

For any hole H of Ω (i.e. H is a bounded connected component of Ω^c), consider a vector field $g \in A_0(\Omega)$ such that $g = \eta$ on ∂H and $g = 0$ elsewhere. Then (3.2) shows that $\int_{\partial H} f_\eta d\sigma = 0$, so we can apply theorem 2.7 (and also remark 2.8 for $n = 2$) to solve the Neumann problem on the complement of Ω with boundary data f_η. Let $\psi \in \mathcal{C}^\infty(\Omega^c) \cap \mathcal{C}(\overline{\Omega^c})$ be a solution such that $\nabla\psi(\infty) = 0$ (this is automatically satisfied for $n > 2$ using the Kelvin transform and computing the derivatives, and can be assumed for $n = 2$ by remark 2.8).

As we did in remark 2.10, it is easily checked that u_g and ψ satisfy the statements of lemma 2.9 if $n \geq 2$, so we deduce from (3.2) that

$$\int_{\partial\Omega} (\varphi - \psi)(\nabla u_g \cdot \eta)d\sigma = 0. \tag{3.3}$$

This equality holds for all u_g harmonic in Ω^c and smooth up to $\partial\Omega$ such that there exists $g \in A_0(\Omega)$ with $u_g = g \cdot \eta$ and $\mathrm{div}g_\tau = \nabla u_g \cdot \eta$ on $\partial\Omega$. We are going to see that (3.3) implies that $\varphi - \psi$ is constant on each connected component of $\partial\Omega$.

Suppose we have a smooth function q on $\partial\Omega$ such that $\int_S q d\sigma = 0$ for each connected component S of $\partial\Omega$. Then, by the maximal de Rham cohomology theorem on the Riemannian manifold S (see [L1], theorem 1.1 of chapter XVIII), the differential form $qd\sigma$ is exact, i.e., there exists a smooth vector field w tangent to S and such that $\mathrm{div}w = q$. On the other hand, the Neumann problem with boundary data q can be solved on Ω^c by theorem

2.7. Therefore, we can assume that this solution (call it u_g) vanishes at infinity. Summarizing, given the function q we have constructed a smooth vector field $w + u_g \cdot \eta \in A_0(\Omega)$ such that $\mathrm{div} w = \nabla u_g \cdot \eta$ on $\partial\Omega$.

So, we deduce from (3.3) that

$$\int_{\partial\Omega} (\varphi - \psi) q d\sigma = 0 \tag{3.4}$$

for all smooth functions q on $\partial\Omega$ such that $\int_S q d\sigma = 0$ for each connected component S of $\partial\Omega$.

Therefore, we deduce from (3.4) that $\varphi - \psi$ is constant on each component of $\partial\Omega$, and we obtain $(\nabla\psi)_\tau = d\varphi = f_\tau$ on $\partial\Omega$. Remember that $\nabla\psi \cdot \eta = f_\eta$, so $\nabla\psi = f$ on $\partial\Omega$. As $f \in \mathcal{C}(\partial\Omega)^n$ and $\Delta\psi = 0$ in Ω^c, we have $\psi \in \mathcal{C}^1(\overline{\Omega^c})$ because each coordinate of $\nabla\psi$ must be given by the Poisson integral of the corresponding coordinate of f. By the Whitney extension theorem (see [Wh], or [Pa] theorem [8]), we can extend ψ inside Ω to have $\psi \in \mathcal{C}^1(\mathbb{R}^n)$. We have finally obtained $\psi \in U_c(\Omega)$ and $\nabla\psi|_{\partial\Omega} = f$, so the theorem is proved. □

Proposition 3.2. *If $\mu \in B(\Omega)^\perp$, then $\mu|_{\partial\Omega} \in bB(\Omega)^\perp$.*

Proof. Let $\nabla\varphi \in B(\Omega)$. We have to check that $\langle \nabla\varphi, \mu|_{\partial\Omega}\rangle = 0$.

Because of $\mu \in B(\Omega)^\perp$,

$$\langle \nabla\varphi, \mu|_{\partial\Omega}\rangle = \int_{\partial\Omega} \nabla\varphi d\mu = \langle \nabla\varphi, \mu\rangle - \int_{\mathrm{int}\Omega} \nabla\varphi d\mu = -\int_{\mathrm{int}\Omega} \nabla\varphi d\mu.$$

Consider a Whitney decomposition $\mathrm{int}\Omega = \bigcup_{i\in\mathbb{N}} Q_i$, where $\{Q_i\}_{i\in\mathbb{N}}$ are disjoint cubes such that $2Q_i \subset \mathrm{int}\Omega$ for all $i \in \mathbb{N}$ and the family $\{2Q_i\}_{i\in\mathbb{N}}$ has finite overlap of order M, and consider a partition of unity subordinated to this decomposition: let $\{\psi_i\}_{i\in\mathbb{N}}$ be a family of $\mathcal{C}^\infty$ functions such that $0 \le \psi_i \le 1$, $\|\nabla\psi_i\| \le C/\mathrm{diam}(Q_i)$ and $\mathrm{supp}\psi_i \subset \frac{3}{2}Q_i$ for each $i \in \mathbb{N}$, so that $\sum_{i\in\mathbb{N}} \psi_i = 1$ in $\mathrm{int}\Omega$. Then $\varphi = \sum_{i\in\mathbb{N}} \psi_i\varphi$ in $\mathrm{int}\Omega$ and at most M terms are non zero in the last sum for all $x \in \mathrm{int}\Omega$.

Observe that

$$\int_{\mathrm{int}\Omega} \nabla(\psi_i\varphi) d\mu = \langle \nabla(\psi_i\varphi), \mu\rangle = 0,$$

because $\mathrm{supp}(\psi_i\varphi) \subset \mathrm{int}\Omega$ and $\nabla(\psi_i\varphi) \in B(\Omega)$. Hence, for $N \in \mathbb{N}$

$$\begin{aligned}
\int_{\mathrm{int}\Omega} \nabla\varphi d\mu &= \int_{\mathrm{int}\Omega} \nabla\varphi d\mu - \sum_{i=1}^N \int_{\mathrm{int}\Omega} \nabla(\psi_i\varphi) d\mu \\
&= \int_{\mathrm{int}\Omega} \nabla\varphi d\mu - \sum_{i=1}^N \int_{\mathrm{int}\Omega} (\varphi\nabla\psi_i + \psi_i\nabla\varphi) d\mu \\
&= \int_{\mathrm{int}\Omega} \nabla\varphi \left(1 - \sum_{i=1}^N \psi_i\right) d\mu + \int_{\mathrm{int}\Omega} \sum_{i=1}^N \varphi\nabla\psi_i d\mu = I + II.
\end{aligned}$$

By dominated convergence, $|I| \to 0$ as $N \to \infty$.

In order to estimate $|II|$, define the set

$$\Omega_N = \{x \in \mathrm{int}\Omega : \exists i > N \text{ with } x \in \mathrm{supp}\psi_i\}.$$

Clearly, $\Omega_{N+1} \subset \Omega_N$ and $\bigcap_{N\in\mathbb{N}} \Omega_N = \emptyset$. Because of the finite overlap of the squares $2Q_i$, we also have

$$\sum_{i=1}^{N} \nabla\psi_i(x) = 0 \quad \text{for all} \quad x \in \text{int}\Omega \setminus \Omega_N. \tag{3.5}$$

Let x_i be the center of Q_i. Using that $\nabla\psi_i \in B(\Omega)$, that supp$\psi_i \subset \text{int}\Omega$ and equation (3.5), we deduce

$$II = \int_{\Omega_N} \sum_{i=1}^{N} (\varphi(x) - \varphi(x_i))\nabla\psi_i(x) d\mu(x).$$

For all $x \in \Omega_N$,

$$\begin{aligned} \left| \sum_{i=1}^{N} (\varphi(x) - \varphi(x_i))\nabla\psi_i(x) \right| &\leq \sum_{i=1}^{N} |\varphi(x) - \varphi(x_i)||\nabla\psi_i(x)| \\ &\leq \sum_{i=1}^{N} \|\nabla\varphi\|_\Omega |x - x_i||\nabla\psi_i(x)|. \end{aligned}$$

If $x \in$ suppψ_i, then $|\nabla\psi_i(x)||x - x_i| \leq C$. Because of the finite overlap of the cubes $2Q_i$, each $x \in \Omega_N$ belongs to the support of at most M functions ψ_i, so the last sum is less than or equal to $CM\|\nabla\varphi\|_\Omega$ (which does not depend on $x \in \Omega_N$). This implies that

$$|II| \leq CM\|\nabla\varphi\|_\Omega |\mu|(\Omega_N).$$

Now, $\lim_{N\to\infty} |\mu|(\Omega_N) = |\mu|(\bigcap_{N\in\mathbb{N}} \Omega_N) = 0$, so $|II| \to 0$ when $N \to \infty$. This completes the proof. $\square$

Theorem 3.3.

$$\kappa_c(\Omega) = \min\left\{ \|\eta d\sigma_o + \mu\|_{\partial_o\Omega} : \mu \in B(\Omega)^\perp, \text{supp}\mu \subset \partial_o\Omega \right\},$$

where $d\sigma_o$ is the surface measure of $\partial_o\Omega$, i.e., the restriction of $d\sigma$ to $\partial_o\Omega$.

Proof. By definition,

$$\kappa_c(\Omega) = \sup\{\langle 1, \Delta\varphi\rangle : \varphi \in U_c(\Omega), \|\nabla\varphi\| \leq 1\}. \tag{3.6}$$

Using Gauss formula, for each $\varphi \in U_c(\Omega)$ we have

$$\begin{aligned} \langle 1, \Delta\varphi\rangle &= \int_{\partial\Omega} \nabla\varphi \cdot \eta d\sigma \\ &= \int_{\partial_o\Omega} \nabla\varphi \cdot \eta d\sigma + \int_{\partial\Omega\setminus\partial_o\Omega} \nabla\varphi \cdot \eta d\sigma = \int_{\partial_o\Omega} \nabla\varphi \cdot \eta d\sigma_o, \end{aligned}$$

because each connected component of $\partial\Omega \setminus \partial_o\Omega$ is the boundary of a smooth bounded domain where φ is harmonic.

Define the functional $\Phi : \mathcal{C}(\Omega)^n \to \mathbb{R}$ as $\Phi(f) = \int_{\partial\Omega} f \cdot \eta d\sigma_o$. Then (3.6) becomes

$$\begin{aligned} \kappa_c(\Omega) &= \sup\{|\Phi(\nabla\varphi)| : \nabla\varphi \in B(\Omega), \|\nabla\varphi\| \leq 1\} \\ &= \sup\{|\Phi(\nabla\varphi)| : \nabla\varphi \in B(\Omega), \|\nabla\varphi\|_\Omega \leq 1\} = \|\Phi\|_{B(\Omega)}, \end{aligned}$$

because $|\nabla\varphi|$ is subharmonic when φ is harmonic, and the maximum principle can be applied.

Clearly, a functional $\widetilde{\Phi}$ on $\mathcal{C}(\Omega)^n$ is an extension of $\Phi|_{B(\Omega)}$ if and only if $\Psi := \widetilde{\Phi} - \Phi$ is orthogonal to $B(\Omega)$, and for all such extensions $\|\Phi\|_{B(\Omega)} \leq \|\widetilde{\Phi}\|$. Hence,

$$\begin{aligned} \|\Phi\|_{B(\Omega)} &= \min\left\{\|\widetilde{\Phi}\| : \widetilde{\Phi} \text{ is an extension of } \Phi|_{B(\Omega)} \text{ to } \mathcal{C}(\Omega)^n\right\} \\ &= \min\left\{\|\Phi+\Psi\| : \Psi \text{ is orthogonal to } B(\Omega)\right\} \\ &= \min\left\{\|\eta d\sigma_o + \mu\|_\Omega : \mu \in B(\Omega)^\perp\right\}, \end{aligned}$$

where we have used by Hahn-Banach's theorem in the first equality, and Riesz's representation theorem 2.1 in the third one.

Observe that for any measure ν supported on Ω,

$$\|\nu|_{\partial\Omega}\|_{\partial\Omega} = \|\nu|_{\partial\Omega}\|_\Omega \leq \|\nu\|_\Omega,$$

so by proposition 3.2,

$$\min\left\{\|\eta d\sigma_o + \mu\|_\Omega : \mu \in B(\Omega)^\perp\right\} = \min\left\{\|\eta d\sigma_o + \mu\|_{\partial\Omega} : \mu \in bB(\Omega)^\perp\right\}.$$

It only remains to check that the last minimum is attained on a measure $\mu \in bB(\Omega)^\perp$ with $\operatorname{supp}\mu \subset \partial_o\Omega$.

By theorem 3.1, for every $\mu \in bB(\Omega)^\perp$ we can find a sequence $h_n \in A_0(\Omega) \cap bB(\Omega)^\perp$ that tends weakly* to μ. Since the connected components of $\partial\Omega$ are a finite number of disjoint compact sets, we see that $h_n\chi_S$ tends weakly* to $\mu|_S$ for every connected component S of $\partial\Omega$. In particular, $h_n\chi_{\partial_o\Omega}$ tends weakly* to $\mu|_{\partial_o\Omega}$ and, by theorem 3.1, $h_n\chi_{\partial_o\Omega} \in bB(\Omega)^\perp$ for all n, so $\mu|_{\partial_o\Omega} \in bB(\Omega)^\perp$. We also clearly have $\|\eta d\sigma_o + \mu|_{\partial_o\Omega}\|_{\partial\Omega} \leq \|\eta d\sigma_o + \mu\|_{\partial\Omega}$.

To summarize, for every measure $\mu \in B(\Omega)^\perp$ with $\operatorname{supp}\mu \subset \partial\Omega$ we have seen that $\mu|_{\partial_o\Omega} \in B(\Omega)^\perp$ and that

$$\|\eta d\sigma_o + \mu|_{\partial_o\Omega}\|_{\partial_o\Omega} = \|\eta d\sigma_o + \mu|_{\partial_o\Omega}\|_{\partial\Omega} \leq \|\eta d\sigma_o + \mu\|_{\partial\Omega},$$

so the theorem is proved. $\square$

4. Some consequences of theorem 3.3

Theorem 4.1. *For every compact set $E \subset \mathbb{R}^n$, $\kappa(E) = \kappa(\partial_o E)$.*

Proof. Since κ is non decreasing set function, $\kappa(E) \geq \kappa(\partial_o E)$.

In order to prove the converse inequality, let $\{\Omega^1_m\}_{m\in\mathbb{N}}$ be a sequence of closures of smooth neighborhoods of E collapsing to E, i.e.

$$E \subset \Omega^1_{m+1} \subset \operatorname{int}\Omega^1_m, \quad \bigcap_{m\in\mathbb{N}} \Omega^1_m = E.$$

Denote by V the unbounded connected component of E^c and consider the bounded open set $V' = \overline{V}^c$. Take an increasing sequence of open sets $\{V'_m\}_m$ such that

$$V'_m \subset V', \quad \overline{V'_m} \subset V'_{m+1}, \quad \partial V'_m \subset \Omega^1_m, \quad \bigcup_{m\in\mathbb{N}} V'_m = V'$$

and with smooth boundary, for all m. Finally, define the sequence

$$\{\Omega^2_m := \Omega^1_m \setminus V'_m\}_{m\in\mathbb{N}}.$$

By construction, the sequence $\{\Omega_m^2\}_{m\in\mathbb{N}}$ is a decreasing sequence of closures of bounded open sets with smooth boundary. Observe also that $\partial_o\Omega_m^1 = \partial_o\Omega_m^2$ for all $m\in\mathbb{N}$.

Now, if $x\in\partial_o E = \partial V$, then $x\in\Omega_m^2$ for all $m\in\mathbb{N}$. On the other hand, if $x\in\Omega_m^2\subset\Omega_m^1$ for all $m\in\mathbb{N}$, then $x\in\overline{V}\cap E$, so $x\in\partial_o E$. Therefore, we have proved that $\bigcap_{m\in\mathbb{N}}\Omega_m^2 = \partial_o E$.

By [Pa] lemma 2.2(7),

$$\begin{aligned}\lim_{m\to\infty}\kappa(\Omega_m^1) &= \kappa(E),\\ \lim_{m\to\infty}\kappa(\Omega_m^2) &= \kappa(\partial_o E).\end{aligned}$$

Let η_m be the outward unit normal vector on $\partial_o\Omega_m^1$ and $d\sigma_m$ the surface measure on $\partial_o\Omega_m^1$. It is easy to see that a measure μ supported in $\partial_o\Omega_m^1$ belongs to $B(\Omega_m^1)^\perp$ if and only if it belongs to $B(\Omega_m^2)^\perp$. Applying theorem 3.3,

$$\begin{aligned}\kappa_c(\Omega_m^1) &= \min\left\{\|\eta_m d\sigma_m+\mu\|_{\partial_o\Omega_m^1} : \mu\in B(\Omega_m^1)^\perp, \mathrm{supp}\mu\subset\partial_o\Omega_m^1\right\}\\ &= \min\left\{\|\eta_m d\sigma_m+\mu\|_{\partial_o\Omega_m^2} : \mu\in B(\Omega_m^2)^\perp, \mathrm{supp}\mu\subset\partial_o\Omega_m^2\right\}\\ &= \kappa_c(\Omega_m^2).\end{aligned}$$

By definition, κ_c is also monotone and $\kappa_c\le\kappa$ and by [Pa] lemma 2.2(1), both capacities coincide on open sets. So, we have

$$\kappa(\Omega_{m+1}^1)\le\kappa(\mathrm{int}\Omega_m^1) = \kappa_c(\mathrm{int}\Omega_m^1)\le\kappa_c(\Omega_m^1) = \kappa_c(\Omega_m^2)\le\kappa(\Omega_m^2).$$

The theorem is proved by letting m tend to infinity. □

Theorem 4.2. *Let f be a real continuous function defined on the cube $Q_0=[0,d]^{n-1}\subset\mathbb{R}^{n-1}$ and let $\Gamma=\{(x,f(x))\in\mathbb{R}^n : x\in Q_0\}$ be the graph of f. Then, there exists a constant $C>0$ depending only on n such that*

$$Cd^{n-1}\le\kappa(\Gamma).$$

Proof. The proof is based on theorem 4.1 and the semiadditivity of κ. Starting from Q_0, consider a decomposition of $\mathbb{R}^{n-1}$ into cubes Q_i of side length d and with disjoint interiors. By doing reflections with respect to the sides of the cubes Q_i, we can extend f to a function $\widetilde{f}$ continuous on $\mathbb{R}^{n-1}$ and such that $\widetilde{f}$ in Q_i is a reflection of f in Q_0.

Let Q^m be a cube in $\mathbb{R}^{n-1}$ of side length md made by the union of m^{n-1} cubes Q_i. Let Γ_m be the graph of the function $\widetilde{f}$ on Q^m, i.e.,

$$\Gamma_m=\{(x,\widetilde{f}(x))\in\mathbb{R}^n : x\in Q^m\},$$

and consider its translation

$$\Gamma_m^t=\{(x,\widetilde{f}(x)+4\|f\|_{Q_0})\in\mathbb{R}^n : x\in Q^m\}.$$

Clearly, the sets Γ_m and Γ_m^t do not intersect. Moreover, they are separated by the set $P_m=\{(x,2\|f\|_{Q_0}) : x\in Q^m\}$.

Finally, let E_m be the region enclosed by Γ_m, Γ_m^t and the $2(n-1)$ pieces of vertical hyperplanes of $\mathbb{R}^n$ that join the endpoints of Γ_m and Γ_m^t. Roughly speaking, E_m is a kind of n-dimensional rectangle which has Γ_m as the

bottom side and Γ_m^t as the top side. By construction, E_m is a compact set that contains P_m, so

$$\kappa(E_m) \geq \kappa(P_m) = C(md)^{n-1} \tag{4.1}$$

by [Pa] lemma 2.2(8). Applying theorem 4.1, the countable semiadditivity of κ and [Pa] lemma 2.2(8), we have

$$\kappa(E_m) = \kappa(\partial_o E_m) \leq C(\kappa(\Gamma_m) + \kappa(\Gamma_m^t) + 20(n-1)\|f\|_{Q_0}(md)^{n-2}). \tag{4.2}$$

By the construction of Γ_m and the countable semiadditivity of κ, $\kappa(\Gamma_m^t) = \kappa(\Gamma_m) \leq Cm^{n-1}\kappa(\Gamma)$, so (4.2) becomes

$$\kappa(E_m) \leq C(2m^{n-1}\kappa(\Gamma) + 20(n-1)\|f\|_{Q_0}(md)^{n-2}). \tag{4.3}$$

Combining (4.1) and (4.3), we get

$$\kappa(\Gamma) \geq \frac{Cm^{n-1}d^{n-1} - C'\|f\|_{Q_0}(md)^{n-2}}{2m^{n-1}} = Cd^{n-1} - C'\frac{\|f\|_{Q_0}d^{n-2}}{m},$$

where $C' > 0$ is an absolute which only depends on the dimension n. Letting $m \to \infty$, we obtain

$$\kappa(\Gamma) \geq Cd^{n-1},$$

and the theorem is proved. □

Remark 4.3. One can show that $\kappa(E) \geq C\text{diam}(E)$ for any continuum $E \subset \mathbb{R}^2$ by using the same ideas as in the proof of theorem 4.2. One starts by choosing two points $a, b \in E$ such that $\text{diam}(E) = |b - a|$ and assuming that these points belong to the real axis in $\mathbb{R}^2$. Then, one extends the set E by symmetries along the real axis as we did before. The rest of the proof remains the same.

The inequality $\kappa(E) \geq C\text{diam}(E)$ for a continuum $E \subset \mathbb{R}^2$ was stated as an open question in problem 2.6 of [Pa] and was first proved by P. Jones by using the notion of curvature of a measure and other capacities called γ_+ and κ_+ (see [Pj], [To2], and [Vo]). We have proved it by a different method.

References

[Ah] L. Ahlfors, *Bounded analytic functions,* Duke Math. J. 14 (1947), pp. 1-11.

[Da] G. David, *Unrectifiable 1-sets have vanishing analytic capacity.* Rev. Mat. Iberoamericana 14 (1998), pp. 369-479.

[DS] N. Dunford and J. T. Schwartz, *Linear operators.* Part I, Interscience, New York and London. MR 22 (1958).

[DU] J. Diestel and J. J. Uhl, Jr., *Vector measures. With a foreword by B. J. Pettis.* Math. Surveys, No. 15. Amer. Math. Soc., Providence R.I. (1977).

[Fo] G. Folland, *Introduction to partial differential equations*, Math. Notes 17, Princeton (1976).

[Ga] T. Gamelin, *Uniform Algebras.* Prentice-Hall, Englewood Cliffs, N.J. (1969).

[Gt] J. Garnett, *Analytic capacity and measure*, Lecture Notes in Mathematics 297, Springer-Verlag, Berlin-New York (1972).

[GK] B. Gustafsson and D. Khavinson, *On annihilators of harmonic vector fields*, Zapiski Nauchn. Semin. POMI 232 (1996), Issled. po Linein. Oper. i Teor. Funktsii. 24, 90-108; English transl. in J. Math. Sci. (New York) 92 (1998), no. 1, pp. 3600-3612.

[L1] S. Lang, *Fundamentals of differential geometry*, Grad. Texts in Math. 191, Springer (1999).

[L2] S. Lang, *Real and functional analysis*, Third Edition. Grad. Texts in Math. 142, Springer (1993).

[Ma] E. Malinnikova, *Measures orthogonal to the gradiens of harmonic functions*, Complex analysis and dynamical systems, Contemp. Math., 364, Amer. Math. Soc., Providence RI (2004), pp. 181-192.

[MC] J.E. Marsden and M. McCracken, *The Hopf bifurcation and its applications*, Applied Math. Sci. 19, Springer (1976).

[MP] P. Mattila and P. V. Paramonov, *On geometric properties of harmonic Lip_1-capacity.* Pacific J. Math. 171 (2) (1995), pp. 469-490.

[Pj] H. Pajot, *Analytic capacity, rectifiability, Menger curvature and the Cauchy integral.* Lecture Notes in Math. 1799, Springer-Verlag, Berlin (2002).

[Pa] P. V. Paramonov, *On harmonic approximation in the $\mathcal{C}^1$ norm*, Mat. Sb. 181 (10) (1990), 1341-1365; English transl. in Math. USSR-Sb. 71 (1) (1992), pp. 183-201.

[Rz] A. Ruiz de Villa, *Estimate of the normal derivative over the boundary of Lipschitz domains.* Int. Math. Res. Not. IMRN (2008), Art. ID rnn 109, 20 pp.

[RT] A. Ruiz de Villa and X. Tolsa, *Characterization and semiadditivity of the C^1-harmonic capacity.* Preprint 2007.

[To1] X. Tolsa, *Painlevé's problem and the semiadditivity of analytic capacity.* Acta Math. 190 (2003), pp. 105-149.

[To2] X. Tolsa, *On the analytic capacity γ_+.* Indiana Univ. Math. J. 51 (2) (2002), pp. 317-344.

[Vi] A. G. Vitushkin, *The analytic capacity of sets in problems of approximation theory,* Uspeikhi Mat. Nauk. 22(6) (1967), pp. 141-199 (Russian); en Russian Math. Surveys 22 (1967), pp. 139-200.

[Vo] A. Volberg, *Calderón-Zygmund capacities and operators on nonhomogeneous spaces.* CBMS Regional Conf. Ser. in Math. 100, Amer. Math. Soc., Providence, RI (2003).

[Wa] F. Warner, *Foundations of differentiable manifolds and Lie groups*, Scott, Foresman, and Company, Glenview, Illinois (1971).

[Wh] H. Whitney, *Analytic extensions of differentiable functions defined in closed sets*, Trans. Amer. Math. Soc. 36 (1934), pp. 63-89.

Departament de Matemàtiques, Universitat Autònoma de Barcelona, Spain
E-mail address: amblesa@mat.uab.cat

Departament de Matemàtiques, Universitat Autònoma de Barcelona, Spain
E-mail address: markmel@mat.uab.cat

Institució Catalana de Recerca i Estudis Avançats (ICREA) and Departament de Matemàtiques, Universitat Autònoma de Barcelona, Spain
E-mail address: xtolsa@mat.uab.cat